Lean complementarity for non-linear magnetostatics

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Lean complementarity aims at giving guaranteed and computable error bounds on the solution of a non-linear magnetostatic problem while attempting to save as much as possible the computational resources required. The main idea is to use an a posteriori error estimate that is able to separate the error components due to the discretization, the linearization and the algebraic error of the iterative linear solver. Once these are available, the stopping criteria for the iterative linear solver, the non-linear solver iterative scheme and the automatic adaptive mesh refinement cycle are all chosen adaptively on the fly. The main idea is that each cycle is stopped once one of the errors becomes predominant and, thus, any reduction of the others does not improve the solution significantly.

Index Terms—**non-linear magnetostatics, Finite Element Method (FEM), adaptive stopping criterion for iterative solvers, adaptive inexact Newton–Raphson, complementarity**

I. Introduction

THIS PAPER extends lean complementarity [1] to the non-linear curl-div system, for example by considering the solution of a magnetostatic paradigm problem. To solve this problem one may use the magnetic scalar potential based formulation using edge elements [2]. That is, the magnetic field h is decomposed as h $= \nabla\Omega + $ h$_s$, where the source magnetic field h$_s$ is represented with edge elements.

Taking strong inspiration from [3] and [4], we devise an a posteriori error estimator that is able to distinguish different error components related to the *discretization* ($\eta_{d,h}$ due to the finite grain of the mesh $\mathcal{M}_h$ and the inaccuracy of the numerical method), the *linearization* ($\eta_{l,h}$ due to the non-linear iterative solver scheme, i.e. the inexact Newton–Raphson or the fixed point) and the *algebraic error* ($\eta_{a,h}$ due to the iterative linear solver). Having the different components in hand, in fact, allows to devise rigorous and adaptive stopping criteria for both linear (in our case an algebraic multigrid solver) and non-linear solver.

II. The novel algorithm

We describe the full algorithm with three flowcharts that represent the three main cycles. Fig. 1 represents the cycle in the variable h where the mesh is adaptively updated by using the local error indicator in each mesh element T. We assume that the initial mesh (i.e. for $k = 0$) is coarse and isotropic given that no error indicator is available yet. This cycle stops when the global guaranteed and computable error estimator for the hth mesh $\mathcal{M}_h$ obtained as the sum of four error components

$$\eta_{tot,h} = \eta_{d,h} + \eta_{l,h} + \eta_{a,h} + \eta_{o,h} \tag{1}$$

is below a user-defined constant ϵ. This is the global stopping criterion of the whole adaptive simulation. $\eta_{o,h}$ is the classical term that takes into account the data oscillation of the sources. The error component e, with $e \in \{d, l, a\}$, is evaluated from local contributions $\eta_{e,T}^{k,i}$ of each element $T \in \mathcal{M}_h$ as

$$\eta_e^{k,i} = \sqrt{\sum_{T \in \mathcal{M}_h} \left(\eta_{e,T}^{k,i}\right)^2}, \tag{2}$$

at the kth iteration of the non-linear solver and at the ith iteration of the linear solver.

Fig. 2 shows the cycle in the variable k which represents the kth iteration of the Newton–Raphson or fixed point scheme. In each iteration of the non-linear solver one has to solve a linear system $\mathbf{A}_h^{k-1} \mathbf{\Omega}_h^k = \mathbf{b}_h^{k-1}$ with, for example, an algebraic multigrid solver. Here, $\mathbf{A}_h^{k-1}$ and $\mathbf{b}_h^{k-1}$ are in general evaluated by using $\mathbf{\Omega}_h^{k-1}$. The non-linear scheme stops the iterations once the discretization error η_d^k dominates with respect to the linearization error η_l^k. The actual threshold for being considered negligible is specified through the constant γ_l set to, for example, 0.1.

Finally, Fig. 3 deals with the solution of the linear system with an iterative solver. First, there is an inner cycle to determine the value of ν as the first integer such that $r_{i+\nu} < \gamma_r \, r_i$ holds, where r_i is the residual of the linear system at the ith iteration and γ_r is again a small constant (for example 0.1). This is required in order to compute the algebraic error estimator $\eta_a^{k,i}$ as proposed in [1].

Second, when a value for ν is set, the three estimators $\eta_a^{k,i}$, $\eta_l^{k,i}$ and $\eta_d^{k,i}$ are computed similarly to [4]. The main novelty w.r.t. [4] is that to compute $\eta_l^{k,i}$ and $\eta_d^{k,i}$ we do not use a flux quasi-equilibration but instead the explicit flux equilibration proposed in [1]. This renders the algorithm simpler and more efficient. The full recipe of how to compute $\eta_l^{k,i}$ and $\eta_d^{k,i}$ is going to be explained in detail in the full paper. The iteration of the linear solver stops when the algebraic error $\eta_a^{k,i}$ becomes negligible with respect to the greater between $\eta_l^{k,i}$ and $\eta_d^{k,i}$.

A. Preliminary numerical results

We tested lean complementarity on the well known Testing in Electromagnetic TEAM problem 13 [5]. Lean complementarity is able to obtain a speed up of a factor of five in the
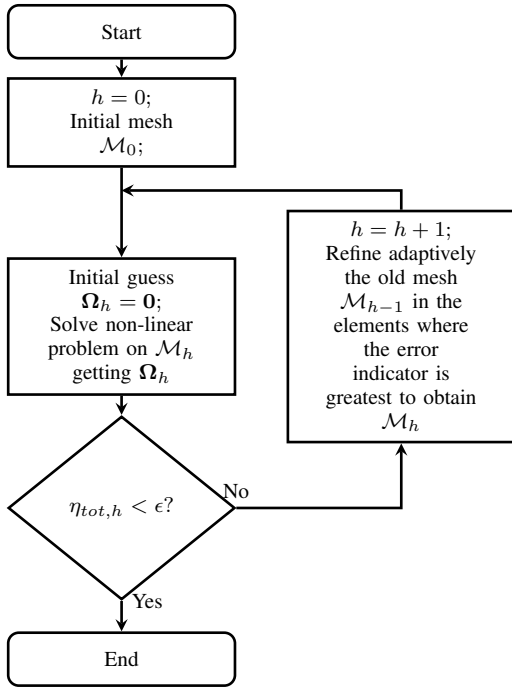
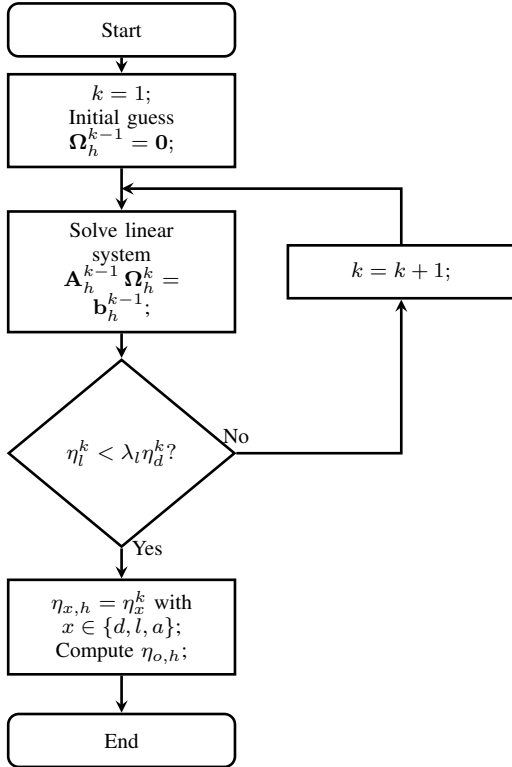Fig. 1. The main cycle in which the mesh is updated adaptively.

Fig. 2. The iterative non-linear solver (for example, fixed point or Newton–Raphson schemes) is inexact and the stopping criterion is chosen adaptively.
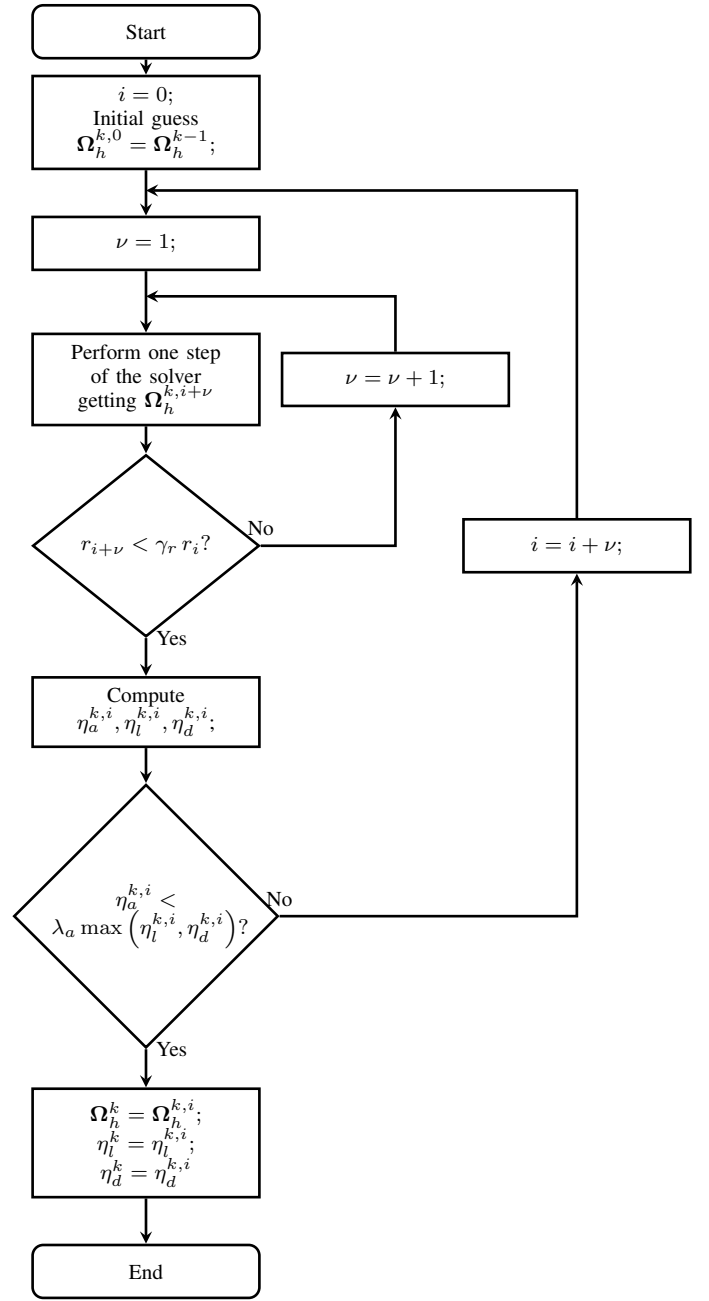

Fig. 3. Solution of the linear system arising at each step of the non-linear solver with an iterative linear solver (for example, an algebraic multigrid solver) with an adaptive stopping criterion.

REFERENCES

[1] R. Specogna, Lean complementarity for Poisson problems, *IEEE Trans. Magn.*, vol. 52, no. 3, 2016.

[2] J.P. Webb, B. Forghani, A single scalar potential method for 3D magnetostatics using edge elements, *IEEE Trans. Magn.*, vol. 25, pp. 4126-4128, 1989.

[3] A. Chaillou, M. Suri, A posteriori estimation of the linearization error for strongly monotone nonlinear operators, *J. Comput. Appl. Math.*, vol. 205, no. 1, pp. 72-87, 2007.

[4] A. Ern, M. Vohralík, Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs, *SIAM J. Sci. Comput.*, vol. 35, no. 4, pp. 1761-1791, 2013.

[5] *TEAM Problem 13: 3-D Non-Linear Magnetostatic Model.* [Online]. Available: http://www.compumag.org/jsite/images/stories/TEAM/ problem13.pdf, accessed Dec. 8, 2016.

total simulation time for a given accuracy with respect to the same simulation with the classical stopping criteria, i.e. both the relative residual of the linear solver and the non-linear iterative solver set to 10^{-8}.